Continuity at the origin

Classify the following functions as either discontinuous at the origin or continuous at the origin. For those functions that are discontinuous, classify them as having a removable or essential discontinuity.

1. $$f(x, y) = \frac{3x^2y}{x^2 + y^2}$$

2. $$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

3. $$f(x, y) = \frac{x^2 - y^2}{x + y}$$

4. $$f(x, y) = \begin{cases} \frac{x^2-y^2}{x-y} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

5. $$f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$$

6. $$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Solution

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. We say that f is continuous at $\mathbf{a}$ if the following conditions are met:

- The function f is defined at $\mathbf{a}$; that is, $f(\mathbf{a})$ exists.
- The limit of $f(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{a}$ exists. Formally, $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$, where L is a real number.
- The limit of the function as $\mathbf{x}$ approaches $\mathbf{a}$ is equal to the value of the function at $\mathbf{a}$. That is, $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.

1. The function does not exist at the point, but the limit exists

$$\lim_{(x,y)\to(0,0)} 3y\frac{x^2}{x^2+y^2} = 0$$

Since it is a function tending to 0 ($(3y)$), multiplied by a bounded function $\frac{x^2}{x^2+y^2}$. To see that this function is bounded:

$$0 \leq x^2$$
$$0 \leq \frac{x^2}{x^2+y^2}$$

Moreover:

$$1 = \frac{x^2+y^2}{x^2+y^2}$$
$$\frac{x^2}{x^2+y^2} \leq \frac{x^2+y^2}{x^2+y^2}$$
$$\frac{x^2}{x^2+y^2} \leq 1$$

Therefore, the function is bounded between 0 and 1. **This is a removable discontinuity.**

2. Calculate the radial limit: $y = mx$

$$L_r = \lim_{x\to 0} \frac{2xmx}{x^2+(mx)^2} = \lim_{x\to 0} \frac{2mx^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{2m}{1+m^2}$$

Since the result depends on m, we can conclude that the limit does not exist. **This is an essential discontinuity.**

3. The function does not exist at the origin. But we can calculate the limit, through the difference of squares:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)(x-y)}{x+y} = \lim_{(x,y)\to(0,0)} x-y = 0$$

This is a removable discontinuity.

4. The function exists at the origin. And we can calculate the limit, through the difference of squares:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)(x-y)}{x-y} = \lim_{(x,y)\to(0,0)} x+y = 0$$

This is a function continuous at the origin.

5. The function does not exist at the origin. But we can calculate the limit:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

Since we are facing a bounded function $\frac{y}{\sqrt{x^2+y^2}}$, multiplied by a function tending to 0 (x). To see that the function is bounded, refer to the mathematical derivation that demonstrates its bounded nature between -1 and 1. **This is a removable discontinuity.**

6. As in the previous case, the limit exists and the function exists at the point **therefore, this is a function continuous at the origin.**